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SPECIAL FUNCTION SOLUTIONS OF THE DIFFUSION EQUATION

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(hib) In this report certain special functions are introduced and their properties discussed. In particular they are shown to be solutions of the diffusion (heat) equation satisfying special initial conditions. These functions are then used to obtain approximate solutions to initial-boundary value problems for the diffusion equation which are particularly accurate for small times.

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I. INTRODUCTION AND PRELIMINARIES

This report is intended as the first of a series in which the problem of heat transfer at a solid boundary will be mathematically investigated with the ultimate objective of developing accurate and efficient numerical algorithms to solve the appropriate governing equations. The basic line of attack will be to use the tools of asymptotic analysis to obtain approximate solutions for equations in the form

$$u_t = a(x)\, u_{xx} + b(x) + c(x)\, u \tag{1.1}$$

valid for small times (in a non-dimensionalized sense to be made more precise). In particular these will be obtained through the use of a "Diffusion Equation Solution Sequence" or DESS, a concept which was introduced in[1]. The approximate solutions will then be incorporated into a numerical scheme which advances over longer time intervals by repeating the basic short-time algorithm as frequently as necessary. A number of problems of engineering concern can be treated using this technique. The problem of particular importance is that of heat transfer in gun barrels. More generally any "transport" equation such as discussed in[2] can be treated in similar fashion.

In the present report we shall not accomplish these longer term objectives but merely wish to lay the foundations by defining and investigating the properties of certain special functions to be denoted $H_\gamma(x,t)$, $H^*_\gamma(x,t)$ and $v_n(x,t)$. These solve the diffusion equation and satisfy certain special initial conditions. In Section VI we shall show how these functions can be effectively utilized for obtaining asymptotic expansions for solutions of initial-boundary value problems for the diffusion equation, valid for small times.

The number of independent variables in diffusion problems can frequently be reduced from two to one by considering the similarity parameter $xt^{-1/2}$, $t>0$. This is true in the present analysis and we have found it convenient not to introduce the functions H^*_γ, H_γ and v_n directly but first to consider the related functions $\overline{H}_\gamma$, $\overline{H}^*_\gamma$ and $\overline{v}_n$ depending only on a single variable z (which may be taken as $x\, t^{-1/2}$). The first two functions are generalizations of the well known repeated integrals of the error function and the last is a related polynomial.

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[1] *J. F. Polk, "Asymptotic Expansions for the Solutions of Parabolic Differential Equations with a small parameter", Ph.D. dissertation, Department of Mathematics, University of Delaware, Newark, DE, 1977.*

[2] *P. J. Roache, "Computational Fluid Dynamics", page 18, 2nd Edition, Hermosa Publishers, Albuquerque, NM, 1976.*

Except for Section VI the contents of this report represent a simplified version of Chapter 1 of the author's doctoral dissertation[1]. Some of the results obtained therein have been re-presented here in report form mainly to provide a conveniently available reference for subsequent work.

The following notational conventions will be used in this report

$$f^*(z) = f(-z)$$

$$f^*(x,t) = f(-x,t)$$

$$f'(z) = \frac{d}{dz} f(z)$$

$$f^{(n)}(z) = \frac{d^n}{dz^n} f(z)$$

$$D_x^n f(x,t) = \frac{\partial^n}{\partial x^n} f(x,t)$$

$$D_t f(x,t) = \frac{\partial}{\partial t} f(x,t)$$

$$[a] = \text{greatest integer} \leq a$$

$$R = (-\infty, \infty) = \text{all real numbers.}$$

Because the gamma function will be frequently encountered in the discussion it is convenient to use the more simplified notation of the factorial function. This is defined by

$$a! = \int_0^\infty t^a e^{-t} dt$$

for $a > -1$ and by

$$a! = \frac{(a+1)!}{a+1}$$

for $a < -1$, $a \neq -2, -3, \ldots$. This relates to the usual gamma function by

$$a! = \Gamma(a+1)$$

The reciprocal of the gamma function is known to be entire with zeros at $0, -1, -2, \ldots$; thus the function $1/a!$ is well defined and finite for all $a \varepsilon R$ with

$$\frac{1}{a!} = 0 \tag{1.2}$$

for a = -1, -2, ... Using factorial notation the well known duplication formula for the gamma function[3] becomes

$$\frac{2^a}{a!} = \frac{\sqrt{\pi}}{((a-1)/2)! \quad (a/2)!} \tag{1.3}$$

II. THE FUNCTIONS $\overline{H}_\gamma$

For any $\gamma \epsilon R$ define the "canonical" jump functions

$$h_\gamma(x) = \begin{cases} 0 & x \leq 0 \\ x^\gamma/\gamma! & x > 0 \end{cases} \tag{2.1}$$

and

$$h^*_\gamma(x) = h_\gamma(-x);$$

for $\gamma = -1$ define

$$\overline{H}_{-1}(z) = (4\pi)^{-1/2} \exp(-z^2/4) \tag{2.2}$$

and for $\gamma > -1$ let

$$\overline{H}_\gamma(z) = \int_R h_\gamma(s)\, \overline{H}_{-1}(s-z)\, ds \tag{2.3}$$

where R = $(-\infty, \infty)$; in particular note that

$$\overline{H}_o(z) = (4\pi)^{-1/2} \int_o^\infty \exp(-(s-z)^2/4) \tag{2.4}$$
$$= \frac{1}{2} \operatorname{erfc}(-z/2).$$

More generally, comparing with equation (7.2.3) of[3], page 299.

$$\overline{H}_n(z) = (4\pi)^{-1/2} \int_o^\infty \frac{s^n}{m!} \exp(-(s-z)^2/4)\, ds \tag{2.5}$$
$$= 2^{n-1}\, i^n \operatorname{erfc}(-z/2)$$

[3] *M. Abramowitz and I. A. Stegun, editors, "Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables", p 256, National Bureau of Standards Applied Mathematics Series #55, Second Edition, 1964.*

Where i^n erfc denotes the n- th integral of the complementary error function. For $\gamma < -1$ we define $\overline{H}_\gamma$ by

$$\overline{H}_\gamma (z) = \overline{H}^{(n)}_{\gamma+n} (z) \tag{2.6}$$

where the superscript denotes differentiation n times and $n = [-\gamma]$ = integer part of $-\gamma$. In order that this definition for $\gamma < -1$ be meaningful it is necessary that $\overline{H}_\gamma$ be differentiable for $-1 \leq \gamma < 0$. This is a consequence of the following proposition.

<u>Proposition 1</u>: The functions $\overline{H}_\gamma$ are entire functions for all $\gamma \varepsilon R$ and

$$\overline{H}^{(n)}_\gamma = \overline{H}_{\gamma-n} \tag{2.7}$$

for any non-negative integer n.

<u>Proof</u>: The function $\overline{H}_{-1}$ is clearly entire. For $\gamma > -1$ consider the related functions

$$g_\gamma (z) = (4\pi)^{1/2} \gamma! \ \exp [z^2/4] \ \overline{H}_\gamma (z)$$

$$= \int_0^\infty s^\gamma \ \exp [- (s^2 - 2sz)/4] \ ds$$

These functions are clearly positive and have n - th derivatives given by

$$g^{(n)}_\gamma [z] = 2^{-n} \int_0^\infty s^{\gamma+n} \ \exp [- (s^2 - 2sz)/4] \ ds$$

$$= 2^{-n} g_{\gamma+n} (z)$$

$$\geq 0.$$

Thus $g_\gamma(z)$ is absolutely monotonic on the real line. Such functions are necessarily analytic[4]. We may conclude that

$$\overline{H}_\gamma (z) = (4\pi)^{-1/2} \exp (-z^2/4) \ g_\gamma(z)$$

is entire for $\gamma > -1$ also. To show that (2.7) holds for n=1 and $\gamma > 0$ we may differentiate (2.3) to obtain

$$\overline{H}'_\gamma (z) = \overline{H}_{\gamma-1} (z)$$

[4] *D. V. Widder, "The Laplace Transform", page 146, Princeton University Press, Princeton, NJ 1941.*

But from (2.2) and 2.4) we see that this must hold for $\gamma = 0$ also and can be extended to all γ using definition (2.6). For $n \geq 2$ equation (2.7) then follows by simple induction. This concludes the proof.

The present version of the argument showing that $\overline{H}_\gamma$ is entire is due to Mr. W. O. Egerland of the Ballistics Research Laboratory[5].

A useful recursive formula can now be obtained. For $\gamma > 0$ we may differentiate (2.3) under the integral sign and obtain

$$\begin{aligned}
\overline{H}_{\gamma-1}(z) &= \overline{H}'_\gamma(z) \\
&= \frac{1}{2} \int_R (s - z)\, h_\gamma(s)\, \overline{H}_{-1}(s - z)\, ds \\
&= \frac{(\gamma+1)}{2} \int_R h_{\gamma+1}(s)\, \overline{H}_{-1}(s-z)\, ds \\
&\quad - \frac{z}{2} \int_R h_\gamma(s)\, \overline{H}_{-1}(s-z)\, ds \\
&= \frac{(\gamma+1)}{2} \overline{H}_{\gamma+1}(z) - \frac{z}{2} \overline{H}_\gamma(z)
\end{aligned}$$

By shifting the index and rearranging terms the last equation can also be written in either of the forms

$$\gamma\, \overline{H}_\gamma(z) = z\, \overline{H}_{\gamma-1}(z) + 2\, \overline{H}_{\gamma-2}(z) \tag{2.8a}$$

$$\overline{H}_\gamma(z) = \frac{(\gamma+2)}{2} \overline{H}_{\gamma+2}(z) - \frac{z}{2} \overline{H}_{\gamma+1}(z). \tag{2.8b}$$

By differentiating these formulas it is easy to extend their validity to all choices of γ.

Let us now derive a power series expansion for $\overline{H}_\gamma(z)$. First note that for $\gamma > -1$, using standard tables of definite integrals[6] we have

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[5] *W. O. Egerland, Private Communication, US Army Ballistic Research Laboratory, Aberdeen Proving Grounds, MD, 1977.*

[6] ——— *, Handbook of Mathematical Tables, Supplement to Handbook of Chemistry and Physics, page 324, Second Edition, The Chemical Rubber Company, Cleveland, OH, 1964.*

$$\bar{H}_\gamma(0) = (4\pi)^{-1/2} \int_0^\infty \frac{s^\gamma}{\gamma!} \exp(-s^2/4)\, ds$$

$$= \frac{2^\gamma (4\pi)^{-1/2}}{\gamma!} \int_0^\infty r^{(\gamma-1)/2} \exp(-r)\, dr$$

$$= \frac{2^\gamma (4\pi)^{-1/2}}{\gamma!} ((\gamma-1)/2)! \qquad (2.9)$$

But then from the duplication formula (1.3)

$$\bar{H}_\gamma(0) = \frac{1}{2(\gamma/2)!} \qquad (2.10)$$

This can be shown to be valid for $\gamma < -1$ also, through the use of recursion formula (2.8). It follows that

$$\bar{H}_\gamma(z) = \sum_{n=o}^{\infty} \bar{H}_\gamma^{(n)}(0)\ z^n / n!$$

$$= \sum_{n=o}^{\infty} \bar{H}_{\gamma-n}(0)\ z^n / n!$$

$$= \frac{1}{2} \sum_{n=o}^{\infty} \frac{z^n}{((\gamma-n)/2)!\ n!} \qquad (2.11)$$

which converges for all z since $\bar{H}_\gamma$ is entire.

A different expansion for $\bar{H}_\gamma(z)$ is useful for $z < 0$ since it turns out that $\bar{H}_\gamma(z)$ vanishes exponentially as $z \to -\infty$. Recall from the proof of Proposition 1 that for $\gamma > -1$

$$\bar{H}_\gamma(z) = \bar{H}_{-1}(z)\ g_\gamma(z) / \gamma!$$

Because the function $g_\gamma(z)$ is entire it may be expressed as a power series. Thus

$$g_\gamma(z) = \sum_{n=o}^{\infty} g_\gamma^{(n)}(0)\ z^n / n!$$

$$= \sum_{n=o}^{\infty} 2^{-n} \, g_{\gamma+n} \, (0) \; z^n \,/\, n!$$

$$= (4\pi)^{-1/2} \sum_{n=o}^{\infty} 2^{-n} \, (\gamma+n)! \;\; \overline{H}_{\gamma+n} \, (0) \; z^n \,/ n!$$

which, for (2.9),

$$= 2^{\gamma} \sum_{n=o}^{\infty} \frac{((\gamma+n-1)/2)!}{n!} \; z^n$$

consequently

$$\overline{H}_{\gamma} \; (z) = \overline{H}_{-1} \, (z) \sum_{n=o}^{\infty} \frac{((\gamma+n-1)/2)!}{\gamma! \;\; n!} \; z^n \tag{2.12}$$

Using induction this formula can be extended to all γ except $\gamma = -1$, -2, -3, ... where the expression for the coefficients becomes indeterminate and requires special interpretation. We shall obtain an alternative formula for these cases presently (equation (2.20) below).

To simplify the evaluation of the functions $\overline{H}_{\gamma}$ note that once $\overline{H}_{\gamma}$ (z) and $\overline{H}_{\gamma-1}$ (z) have been determined for some γ then the recursive formula (2.8) can be used to evaluate $\overline{H}_{\gamma+n}$ (z) for any integer n. In particular we can write

$$\overline{H}_{\gamma+n} \, (z) = p_{\gamma,n} \, (z) \; \overline{H}_{\gamma} \, (z) + q_{\gamma,n} \, (z) \; \overline{H}_{\gamma-1} \, (z) \tag{2.13}$$

where $p_{\gamma,n}$ (z) and $q_{\gamma,n}$ (z) are polynomials of degree $|n+1|$ and $|n|-1$ respectively which satisfy the following recursion formulas

$$p_{\gamma,o} \, (z) = 1 \qquad\qquad q_{\gamma,o} \, (z) = 0$$

$$p_{\gamma,-1} \, (z) = 0 \qquad\qquad q_{\gamma,-1} \, (z) = 1$$

$$(\gamma+n) \; p_{\gamma,n}(z) = z \; p_{\gamma,n-1} \, (z) + 2 \; p_{\gamma,n-2} \, (z) \tag{2.14}$$

$$(\gamma+n) \; q_{\gamma,n} \, (z) = z \; q_{\gamma,n-1} \, (z) + 2 \; q_{\gamma,n-2} \, (z) \tag{2.15}$$

These may be verified without difficulty using induction and we will not include the details here. The first few polynomials are

$$p_{\gamma,-4}(z) = \frac{1}{8}\gamma z^2 + \frac{1}{4}\gamma(\gamma-2) \qquad q_{\gamma,-4}(z) = -\frac{1}{8}z^3 - \frac{1}{4}(2\gamma-3)\ z$$

$$p_{\gamma,-3}(z) = -\frac{1}{4}\gamma z \qquad q_{\gamma,-3}(z) = \frac{1}{4}z^2 + \frac{1}{2}(\gamma-1)$$

$$p_{\gamma,-2}(z) = \frac{1}{2}\gamma \qquad q_{\gamma,-2}(z) = -\frac{1}{2}z$$

$$p_{\gamma,-1}(z) = 0 \qquad q_{\gamma,-1}(z) = 1$$

$$p_{\gamma,0}(z) = 1 \qquad q_{\gamma,0}(z) = 0 \tag{2.16}$$

$$p_{\gamma,1}(z) = \frac{z}{\gamma+1} \qquad q_{\gamma,1}(z) = \frac{2}{(\gamma+1)}$$

$$p_{\gamma,2}(z) = \frac{z^2 + 2(\gamma+1)}{(\gamma+1)(\gamma+2)} \qquad q_{\gamma,2}(z) = \frac{2z}{(\gamma+1)(\gamma+2)}$$

$$p_{\gamma,3}(z) = \frac{z^3 + 2\ (2\gamma+3)\ z}{(\gamma+1)(\gamma+2)(\gamma+3)} \qquad q_{\gamma,3}(z) = \frac{2z^2 + 4\ (\gamma+2)}{(\gamma+1)(\gamma+2)(\gamma+3)}$$

Of particular interest are the functions $\overline{H}_n(z)$ where n is an integer. Choosing $\gamma=0$ in (2.13) yields.

$$\overline{H}_n(z) = p_{o,n}(z)\ \overline{H}_o + q_{o,n}(z)\ \overline{H}_{-1} \tag{2.17}$$

where from (2.2) and (2.4)

$$\overline{H}_o = \frac{1}{2}\,\mathrm{erfc}\,(-z/2)$$

and

$$\overline{H}_{-1} = (4\pi)^{-1/2}\exp(-z^2/4)$$

Thus from (2.16)

$$\overline{H}_{-4}(z) = (-1/2)^3\ (z^3 - 6z)\ \overline{H}_{-1}$$

$$\overline{H}_{-3}(z) = (1-1/2)^2\ (z^2 - 2)\ \overline{H}_{-1}$$

$$\overline{H}_{-2}(z) = (-1/2)\ z\ \overline{H}_{-1}$$

$$\overline{H}_1(z) = z\,\overline{H}_o + 2\,\overline{H}_{-1} \tag{2.18}$$

$$\overline{H}_2(z) = \left[(z^2+2)\,\overline{H}_o + 2z\,\overline{H}_{-1}\right]/2$$

$$\overline{H}_3(z) = \left[(z^3+6z)\,\overline{H}_o + 2\,(z^2+4)\,\overline{H}_{-1}\right]/3!$$

$$\overline{H}_4(z) = \left[(z^4+12z^2+12)\,\overline{H}_o + 2\,(z^3+10z)\,\overline{H}_{-1}\right]/4!$$

The cases n= -1, -2 simplify since it can be shown using (2.14) that

$$p_{o,n}(z) = 0 \tag{2.19}$$

for $n \le -1$ and thus from (2.17)

$$\overline{H}_n(z) = q_{o,n}(z)\,\overline{H}_{-1}(z) \tag{2.20}$$

n= -1, -2, ... For $|n| \le 12$ the polynomials $p_{o,n}$ and $q_{o,n}$ are listed explicitly in the Appendix.

The asymptotic behavior of $\overline{H}_n(z)$ as $|z| \to \infty$ is completely determined for n= -1, -2, by the representation (2.20). The behavior of $\overline{H}_\gamma(z)$ for $\gamma = -1, -2, \ldots$ is somewhat different, especially for $z \to +\infty$. We conclude this section by stating without proof[1] two propositions which characterize the asymptotic properties of $\overline{H}_\gamma$.

Proposition 2: For any interger $N \ge 0$ and any $\gamma \in R$ with $\gamma = -1, -2, \ldots$ there exists a constant $K_{\gamma,N} \ge 0$ such that

$$\overline{H}_\gamma(z) = (-2/z)^{\gamma+1}\,\overline{H}_{-1}(z)\left[\sum_{n=o}^{N} \frac{(\gamma+2n)!}{\gamma!\;n!}\,(-z^2)^n + R_{\gamma,N}(z)\right] \tag{2.21}$$

where $R_{\gamma,N}(z)$ is a remainder term satisfying

$$|R_{\gamma,N}(z)| \le K_{\gamma,N}\,|z|^{-2(N+1)}$$

uniformly for $z < 0$.

Proposition 3: For any integer $n \geq 0$, $\gamma \in R$ and $z_o > 0$ there exists a constant $K_{\gamma,n,z_o} \geq 0$ such that

$$\overline{H}_\gamma (z) = \sum_{n=o}^{\infty} \frac{z^{\gamma-2n}}{\gamma!\,(\gamma-2n)!} + \overline{R}_{\gamma,n} (z) \tag{2.22}$$

where $\overline{R}_{\gamma,n} (z)$ is a remainder term satisfying

$$|\overline{R}_{\gamma,n} (z)| \leq K_{\gamma,n,z_o} \; z^{\gamma-2n-2}$$

uniformly for $z \geq z_o$.

Note that Proposition 3 remains valid when $\gamma = -1,-2,\ldots$ although it is somewhat degenerate in these cases since the summation term vanishes in view of (1.2). Proposition 2 can also be shown to hold when γ is a negative integer by properly interpreting the indeterminate expression

$$\frac{(\gamma + 2n)!}{\gamma!}$$

III. THE FUNCTIONS $\overline{H}^*_\gamma$ and $\overline{v}_n$

From the notation introduced in Section I we have

$$\overline{H}^*_\gamma (z) = \overline{H}_\gamma (-z) \tag{3.1}$$

for all $\gamma \in R$. From (2.7) and (2.8) it follows immediately that for any $\gamma \in R$

$$\overline{H}^{*\,(n)}_\gamma (z) = (-1)^n \, \overline{H}^*_{\gamma -n} (z) \tag{3.2}$$

$$\gamma\overline{H}^*_\gamma (z) = -z \, \overline{H}^*_{\gamma-1} (z) + 2 \, \overline{H}^*_{\gamma-2} (z) \tag{3.3a}$$

and

$$\overline{H}^*_\gamma\ (z) \quad = \quad ((\gamma+2)/2)\ \overline{H}^*_{\gamma+2}\ (z) + \frac{z}{2}\overline{H}^*_{\gamma+1}\ (z) \tag{3.3b}$$

From (2.2) it is clear that $\overline{H}_{-1} = \overline{H}^*_{-1}$ and thus from (3.2) and (2.7) we have

$$\overline{H}^*_n\ (z) \quad = (-1)^{n+1}\ \overline{H}_n\ (z) \tag{3.4}$$

for $n = -1,-2,-3,\ldots$

The functions $\overline{v}_n$ are defined for integer values of n by

$$\overline{v}_n\ (z) \quad = \int_R \frac{s^n}{n!}\ \overline{H}_{-1}\ (s-z)\ dz \tag{3.5}$$

Comparing with (2.3) we obtain the identity

$$\overline{v}_n\ (z) \quad = \overline{H}_n\ (z) + (-1)^n\ \overline{H}^*_n\ (z) \tag{3.6}$$

which, from (3.4), gives

$$\overline{v}_n\ (z) = 0 \tag{3.7}$$

for $n = -1,-2,-3,\ldots$; from (2.7) and (3.2) we see that

$$\overline{v}_n^{(k)}\ (z) = \overline{v}_{n-k}\ (z) \tag{3.8}$$

and from (2.8a) and (3.3a) we have

$$n\ \overline{v}_n\ (z) \quad = \quad z\ \overline{v}_{n-1}\ (z) + 2\ \overline{v}_{n-2}\ (z) \tag{3.9}$$

Evaluating the integral in (3.5) explicitly yields

$$\begin{aligned} \overline{v}_o\ (z) &= 1 \\ \overline{v}_1\ (z) &= z \end{aligned} \tag{3.10}$$

But then, comparing (3.9) with (2.14) and (3.10) with (2.16) we see that the functions $\overline{v}_n$ coincide with the polynomials $p_{o,n}$

$$\overline{v}_n\ (z) \quad = p_{o,n}\ (z) \tag{3.11}$$

for any n. Moreover the recursion relation (3.9) can be used to verify the following representation for $\bar{v}_n$

$$\bar{v}_n(z) = \sum_{k=o}^{[n/2]} \frac{z^{n-2k}}{k!\,(n-2k)!} \tag{3.12}$$

n = 0, 1, 2, ... where [n/2] = greatest integer $\leq n/2$. For $n \leq 12$ these are listed explicitly in the Appendix. Equation (3.10) can also be inverted to obtain

$$\frac{z^n}{n!} = \sum_{k=o}^{[n/2]} \frac{(-1)^k\, \bar{v}_{n-2k}(z)}{k!} \tag{3.13}$$

IV. THE FUNCTIONS H_γ

In this section the relationship between the functions $\bar{H}_\gamma$ discussed in Section II and the diffusion equation is made clear by introducing the functions H_γ. These are defined for any $\gamma \varepsilon R$ by

$$H_\gamma(x,t) = \sqrt{t}^{\gamma}\, \bar{H}_\gamma(x/\sqrt{t}) \tag{4.1}$$

for t > 0, with

$$H_\gamma(x,0) = h_\gamma(x) \tag{4.2}$$

for t = 0. Specifically for $\gamma = -1$ and $\gamma = 0$ we have

$$H_{-1}(x,t) = (4\pi t)^{-1/2} \exp(-x^2/4t) \tag{4.3}$$

$$H_o(x,t) = \frac{1}{2} \operatorname{erfc}(-x/2\sqrt{t}) \tag{4.4}$$

For $\gamma > -1$ these functions can also be given an integral representation using (2.3)

$$\begin{aligned} H_\gamma(x,t) &= \int_R h_\gamma(s)\, H_{-1}(s-x,t)\, ds \\ &= (4\pi t)^{-1/2} \int_0^\infty \frac{s^\gamma}{\gamma!} \exp\left[-(s-x)^2/4t\right] ds \end{aligned} \tag{4.5}$$

The assignment of initial values h_γ in (4.2) is not arbitrary but provides a continuous extension of H_γ from $t > 0$ into $t \geq 0$, except possibly at the point $x = 0$, $t = 0$. To see this note from (2.21) that for $x < 0$

$$\lim_{t\to o} H_\gamma (x,t) = \lim_{t\to o} \sqrt{t}^\gamma \, \bar{H}_\gamma (x/\sqrt{t})$$

$$= \lim_{t\to o} \sqrt{t}^\gamma \exp (-x^2/4t) (-\sqrt{4t}/x)^{\gamma+1} \left[\frac{1}{m!} + 0 (t/x^2)\right]$$

$$= 0$$

and from (2.22) for $x > 0$

$$\lim_{t\to o} H_\gamma(x,t) = \lim_{t\to o} \frac{\sqrt{t}^\gamma (x/\sqrt{t})^\gamma}{\gamma!} [1 + 0(t/x^2)]$$

$$= \frac{x^\gamma}{\gamma!} .$$

The functions H_γ can be shown to be continuous at (0,0) if and only if $\gamma>0$ and bounded if and only if $\gamma \geq 0$. When γ is a negative integer the initial values (4.2) are seen to vanish because of (1.2). This does not fully convey the limiting behavior of H_γ as $t\to o$ since this can be properly expressed only in terms of distributions or generalized functions. A more accurate formulation would be

$$H_{-n} (x,0) = \delta^{(n-1)} (x)$$

for $n = 1,2,3, \ldots$ where $\delta (x)$ is the usual Dirac delta function and $\delta^{(k)}$ indicates its k-th generalized derivative. Since these concepts will not be required in our analysis we shall not discuss them further.

Evaluation of H_γ for the important cases where γ is an integer can be accomplished using (2.17) from which we have

$$H_n (x,t) = \sqrt{t}^n \left[p_{0,n}(x/\sqrt{t}) \, H_0(x,t) + \sqrt{t} \, q_{o,n} (x/\sqrt{t}) \, H_{-1}(x,t)\right] \quad (4.6)$$

where the polynomials $p_{o,n}$ and $q_{o,n}$ are given in the Appendix. In particular the first few of these are

$$H_{-5} = \left[(x^4 - 12x^2 t + 12t^2) / (-2t)^4\right] H_{-1}$$

$$H_{-4} = \left[(x^3 - 6xt) / (-2t)^3\right] H_{-1}$$

$$
\begin{aligned}
H_{-3} &= \left[(x^2 - 2t) / (-2t)^2\right] H_{-1} \\
H_{-2} &= -(x/2t)\, H_{-1} \\
H_1 &= x H_o + 2t H_{-1} \\
H_2 &= \left[(x^2 + 2t) H_o + 2xt H_{-1}\right] /2 \\
H_3 &= \left[(x^3 + 6xt) H_o + 2(x^2 t + 4t^2) H_{-1}\right] /3! \\
H_4 &= \left[(x^4 + 12x^2 t + 12t^2) H_o + 2(x^3 t + 10xt^2) H_{-1}\right] /4!
\end{aligned}
\tag{4.7}
$$

The properties of the function H_γ follow directly from those of the functions $\overline{H}_\gamma$. Setting $x = 0$ in (4.1) and using (2.10) we see

$$H_\gamma (0,t) = \frac{\sqrt{t}^{\gamma}}{2(\gamma/2)!} = \frac{1}{2} h_{\gamma/2} (t), \quad t \geq 0. \tag{4.8}$$

Differentiating (4.1) and recalling (2.7) and (2.8) yields the formulas

$$D_t H_\gamma = H_{\gamma-2} \qquad t > 0 \tag{4.9}$$

$$D_x H_\gamma = H_{\gamma-1} \qquad t > 0 \tag{4.10}$$

and

$$D_x^n H_\gamma = H_{\gamma-n} \qquad t > 0 \tag{4.11}$$

It immediately follows that H_γ is a solution to the diffusion equation

$$\left[D_t - D_x^2\right] H_\gamma (x,t) = 0 \tag{4.12}$$

in the region $t > 0$. More generally for any $a > 0$ and any x_o the function $H_\gamma (x-x_o, at)$ satisfies the equation

$$\left[D_t - aD_x^2\right] H_\gamma (x-x_o, at) = 0 \tag{4.13}$$

for $t > 0$ and has the initial values

$$H_\gamma (x-x_o, 0) = h_\gamma (x-x_o). \tag{4.14}$$

From the recursion formulas (2.12) we obtain

$$\gamma H_\gamma = x H_{\gamma-1} + 2t H_{\gamma-2} \tag{4.15a}$$

$$H_\gamma = (1/2t) \left[(\gamma+2) H_{\gamma+2} - X H_{\gamma+1}\right] \tag{4.15b}$$

for any $\gamma \varepsilon R$.

A number of bounds on the growth of the functions H_γ have been established in[1] but for present purposes we need only the following: For any $x_o < 0$ and any $T > 0$ there exists a constant $K \geq 0$ such that

$$|H_\gamma (x,t)| \leq K \exp(-x^2/4t) \tag{4.16}$$

uniformly for all $x \leq x_o$ and $0 \leq t \leq T$. This can be verified by noting from (2.21) that as $x^2/t \to \infty$

$$H_\gamma (x,t) = \sqrt{t}^\gamma \; \bar{H}_\gamma (x/\sqrt{t})$$

$$= (4\pi)^{-1/2} \sqrt{t}^\gamma (-2\sqrt{t}/x)^{\gamma+1} \exp(-x^2/4t) \left[1 + 0(t/x^2) \right]$$

and $-x/\sqrt{t} \geq -x_o/\sqrt{t} > 0$ in the region of interest.

V. THE FUNCTIONS H^*_γ and v_n

The functions H^*_γ are defined for $\gamma \varepsilon R$ by

$$H^*_\gamma (x,t) = H_\gamma (-x,t), \qquad t \geq 0 \tag{5.1}$$

From the properties of H_γ we immediately have

$$H^*_\gamma (x,0) = h^*_\gamma (x) \tag{5.2}$$

$$H^*_\gamma (0,t) = \frac{1}{2} h_{\gamma/2} (t) \qquad t \geq 0 \tag{5.3}$$

$$D_t \; H^*_\gamma = H^*_{\gamma-2} \qquad t > 0 \tag{5.4}$$

$$D_x \; H^*_\gamma = -H^*_{\gamma-1} \qquad t > 0 \tag{5.5}$$

$$D^n_x \; H^*_\gamma = (-1)^n H^*_{\gamma-n} \tag{5.6}$$

$$\left[D_t - aD^2_x \right] H^*_\gamma (x-x_o, at) = 0 \qquad a>0, x\varepsilon R, t>0 \tag{5.7}$$

$$\gamma H^*_\gamma = -x H^*_{\gamma-1} + 2t \; H^*_{\gamma-2} \qquad t \geq 0 \tag{5.8a}$$

$$H^*_\gamma = (1/2t) \left[(\gamma+2) \; H^*_{\gamma+2} + x H^*_{\gamma+1} \right] \qquad t \geq 0. \tag{5.8b}$$

From (3.4) we have

$$H_n^* = (-1)^{n+1} H_n \tag{5.9}$$

for n = -1, -2, ...

The functions v_n are defined for integer values of n by

$$v_n(x,t) = \sqrt{t}^{\,n} \; \bar{v}_n(x/\sqrt{t}) \tag{5.10}$$

for t > 0, with

$$v_n(x,0) = x^n/n! \tag{5.11}$$

for t = 0. Using (3.5), (3.6) and (3.12) we see that these may also be written in the forms

$$v_n(x,t) = \int_R \frac{s^n}{n!} H_{-1}(x-s,t)\, ds \tag{5.12}$$

$$v_n(x,t) = H_n(x,t) + (-1)^n H_n^*(x,t) \tag{5.13}$$

and

$$v_n(x,t) = \sum_{k=o}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!} \frac{t^k}{k!} \tag{5.14}$$

For $n \geq 0$ these functions are thus identical (except for a factor n!) with the well known heat polynomials discussed in Ref 7 and 8; they are fundamentally important because they are the polynomial solutions of the diffusion equation

$$\left[D_t - D_x^2\right] v_n(x,t) = 0 \tag{5.15}$$

with the polynomial initial values (5.11). Evaluation along x = 0 clearly yields

$$v_n(0,t) = \begin{cases} \dfrac{t^{n/2}}{(n/2)!} & \text{if } n = 0,2,4, \ldots \\ 0 & \text{if } n = 1,3,5, \ldots . \end{cases} \tag{5.16}$$

For $n \leq -1$ note from (3.7) that

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[7] *P. C. Rosenbloom and D. V. Widder, "Expansions in Terms of Heat Polynomials and Related Functions," Transactions of the American Mathematical Society, 92 (1959), pp. 220-266.*

[8] *D. V. Widder, "The Heat Equation," Academic Press, New York, 1975.*

$$v_n(x,t) = 0 \tag{5.17}$$

identically, for all x and t.

VI. INITIAL-BOUNDARY VALUE PROBLEMS

As indicated in the introduction we shall eventually use the functions H_γ, H^*_γ and v_n to develop asymptotic expansions, accurate for small times, for solutions to equations such as (1.1). This will be accomplished in later reports. We are already in a position to do this for the diffusion equation itself, however, and shall now show how various boundary and initial conditions can be effectively handled.

For illustrative purposes let us first consider two simple but important problems for the semi-infinite domain $x \geq 0$, $t \geq 0$

$$\begin{aligned} \left[D_t - aD_x^2\right] u &= 0 && x,\ t > 0 \\ u(x,0) &= h_\gamma(x-x_o) && x > 0 \\ u(0,t) &= 0 && t \geq 0 \end{aligned} \tag{BVP$_1$}$$

and

$$\begin{aligned} \left[D_t - aD_x^2\right] w &= 0 && x,t > 0 \\ w(x,0) &= 0 && x \geq 0 \\ w(0,t) &= h_\gamma(t) && t > 0. \end{aligned} \tag{BVP$_2$}$$

These problems have the exact solutions

$$u(x,t) = H_\gamma(x-x_o,\ at) - H^*_\gamma(x+x_o,\ at) \tag{6.1}$$

and

$$w(x,t) = 2a^{-\gamma}\ H^*_{2\gamma}(x,at) \tag{6.2}$$

respectively, as can be verified by noting (4.13) and (5.7) and by directly substituting (4.14), (5.2) and (5.3) into (6.1) and (6.2). We can also guarantee that these solutions are unique by imposing additional growth conditions on u and w but this is not crucial to our discussion. One important point to note regarding these functions is that they are defined not just for $x \geq 0$ but for all x. Thus, they can also be considered as solutions to the following Cauchy or pure initial value problems respectively:

$$\left[D_t - aD_x^2\right] u = 0$$

$$u\ (x,0) = h_\gamma\ (x-x_o) - h^*_\gamma\ (x+x_o) \qquad (IVP)_1$$

and

$$\left[D_t - aD_x^2\right] w = 0$$

$$w\ (x,o) = 2a^{-\gamma}\ h^*_{2\gamma}\ (x) \qquad (IVP)_2$$

This equivalence between initial-boundary value problems and Cauchy problems is a basic property of the diffusion equation; it can also be used for dealing with the more complicated problems involving a bounded domain $0 \leq x \leq \ell$ as we now intend to show.

Consider the general initial-boundary value problem

$$\left[D_t - aD_x^2\right] u = 0 \qquad 0 < t \leq T,\ 0 < x < \ell \qquad (6.3)$$

$$u\ (x,0) = f(x) \qquad 0 \leq x \leq \ell \qquad (6.4)$$

$$u\ (0,t) = g_o\ (t) \qquad 0 < t \leq T \qquad (6.5)$$

$$u\ (\ell,t) = g_1\ (t) \qquad 0 < t \leq T \qquad (6.6)$$

It may be supposed that the functions $f(x)$, $g_o(t)$ and $g_1(t)$ arise from thermocouple measurements and do not have any specific "analytic" form. However, by using a combination of polynomials and jump functions h_γ, they can usually be effectively approximated with relatively few terms. For instance we may have the initial and boundary conditions

$$f(x) = \sum_{k=o}^{n} \left[a_k x^k + \sum_{j=o}^{n} b_{jk}\ h_k\ (x-x_j)\right] + e(x)$$

and

$$g_o\ (t) = 0 = g_1\ (t)$$

where $e(x)$ is an error term combining the errors in measurement and the errors in representation which is bounded in the form

$$|e\ (x)| \leq e_o \qquad 0 \leq x \leq \ell$$

for some "small" number $e_o > 0$. Using linearity the solution to problem (6.3) - (6.6) can then be written in the form

$$u(x,t) = \sum_{k=o}^{n}\left[a_k\, u_k(x,t) + \sum_{j=o}^{n} b_{jk}\, w_{jk}(x,t)\right] + E(x,t)$$

where u_k and w_{jk} denote solutions to (6.3) which vanish along $x = 0$ and $x = \ell$ and satisfy

$$u_k(x,0) = x^k \qquad 0 \; x$$

$$w_{jk}(x,0) = h_k(x-x_j)$$

respectively. The term E is an error term which will also satisfy the bound

$$|E(x,t)| \le e_o$$

because of the maximum principle for parabolic differential equations. Similar comments apply to problem (6.3) - (6.6) when f(x) vanishes and the boundary values $g_o(t)$ or $g_1(t)$ are non-zero. We are thus led to consider the following two special forms of problem (6.3) - (6.6)

$$\begin{aligned}
\left[D_t - aD_x^2\right] u &= 0 && 0 < t \le T,\ 0 < x < \ell \\
u(x,0) &= h_n(x-x_o) && 0 \le x,\ x_o \le \ell \\
u(0,t) &= 0 && 0 < t \le T \\
u(\ell,t) &= 0 && 0 < t \le T
\end{aligned} \qquad (BVP)_3$$

and

$$\begin{aligned}
\left[D_t - aD_x^2\right] w &= 0 && 0 < t \le T,\ 0 < x < \ell \\
w(x,0) &= 0 && 0 \le x \le \ell \\
w(0,t) &= h_\gamma(t) && 0 < t \le T \\
w(\ell,t) &= 0 && 0 < t \le T
\end{aligned} \qquad (BVP)_4$$

In $(BVP)_3$ we require that n is a non-negative interger; in $(BVP)_4$ γ can be any non-negative real number. A solution to these problems can be obtained by defining an extension of the initial values to the entire real line in such a way that the solution of the resulting Cauchy problem must also satisfy the correct boundary conditions along $x = 0$ and $x = \ell$

For problem $(BVP)_3$ consider the following periodic extension of the given initial values:

$$\bar{f}(x) = \begin{cases} h_n(x - 2k\ell - x_o), & 2k\ell \le x \le (2k+1)\ell \\ -h_n^*(x - (2k+2)\ell + x_o), & (2k+1)\ell \le x < (2k+2)\ell \end{cases}$$

where $k = 0, \pm 1, \pm 2, \ldots$. This function repeats with a period 2ℓ and is antisymmetric about any of the points $x = k\ell$, where $k = 0, \pm 1, \pm 2, \ldots$. A single cycle of $\bar{f}$ is indicated in Figure 1 over the interval $[0, 2\ell]$. Since $\bar{f}$ is antisymmetric about $x = 0$ and $x = \ell$ in particular then the solution of the Cauchy problem

$$\left[D_t - D_x^{\,2}\right] u = 0$$
$$u(x, 0) = \bar{f}(x) \qquad (IVP)_3$$

will also be antisymmetric about $x = 0$ and $x = \ell$ and must therefore be a solution to $(BVP)_3$ as well. To obtain the solution to $(IVP)_3$ in a useful form note first, that in the interval $[0, 2\ell]$ the function $\bar{f}$ can also be written as the sum of jump functions in the form

$$\bar{f}(x) = h_n(x-x_o) + \sum_{k=o}^{n} f_k h_k(x-\ell) + (-1)^n h_n(x-2\ell + x_o) \qquad (6.7)$$

$o \le x \le 2\ell$, where f_k denotes the jump in the k-th derivative values of $\bar{f}$ at $x = \ell$, namely

$$f_k = -(1 + (-1)^k) \frac{(\ell - x_o)^{n-k}}{(n-k)!} \quad . \qquad (6.8)$$

By adding on all other jumps occuring to the right and left of $[0,2\ell]$ in proper sequence we obtain

$$\bar{f}(x) = \sum_{j=o}^{\infty}\Big[h_n(x-2j\ell-x_o) + \sum_{k=o}^{n} f_k h_k((x-(2j+1)\ell)$$
$$+ (-1)^n h_n(x-(2j+2)\ell + x_o)\Big] \qquad (6.9)$$
$$- \sum_{j=o}^{\infty}\Big[h_n^*(x+2j\ell + x_o) + \sum_{k=o}^{n} f_k h_k^*(x+(2j+1)\ell)$$
$$+ (-1)^n h_n^*(x + (2j+2)\ell - x_o)\Big]$$

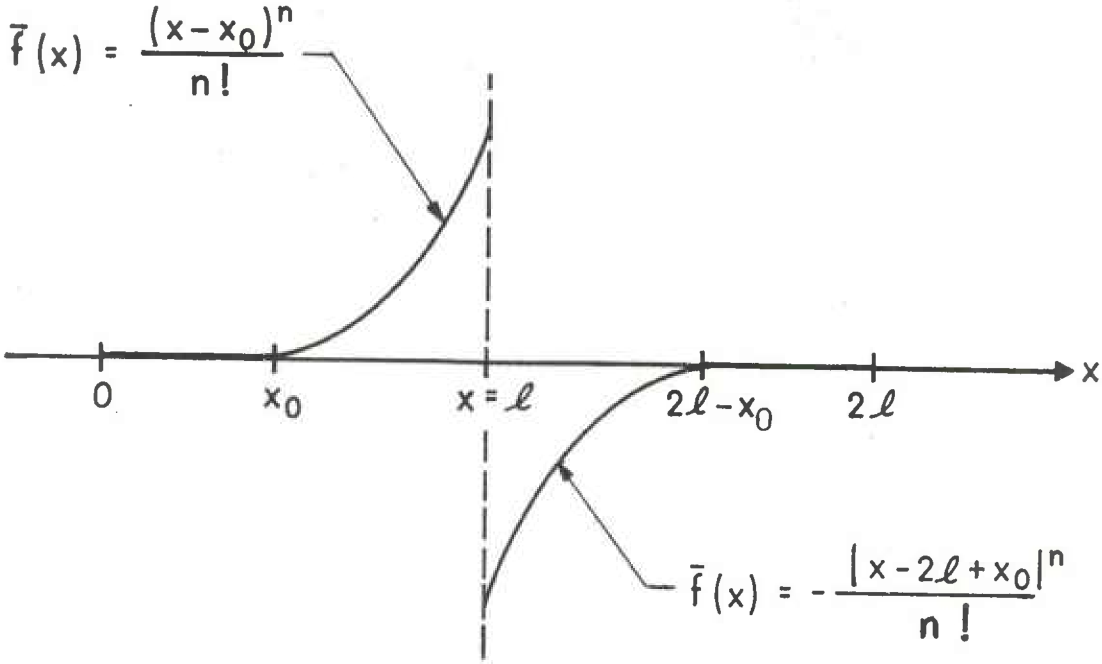

Figure 1. Graph of $\bar{f}(x)$ in the interval $[0,2\ell]$.

The solution to $(IVP)_3$ can thus be written (formally at least)

$$u(x,t) = \sum_{j=o}^{\infty}\Big[H_n\ (x-2j\ell-x_o,at) + \sum_{k=o}^{n} f_k\ H_k\ (x-(2j+1)\ell,at)$$
$$+ (-1)^n\ H_n\ (x-(2j+2)\ell+x_o,at)\Big]$$
$$- \sum_{j=o}^{\infty}\Big[H^*_n\ (x+2j\ell+x_o,at) + \sum_{k=o}^{n} f_k\ H^*_k\ (x+(2j+1)\ell,at)$$
$$+ (-1)^n\ H^*_n\ (x+(2j+2)\ell-x_o,at)\Big] \qquad (6.10)$$

We denote the truncation of this to N terms by u_N

$$u_N(x,t) = \sum_{j=o}^{N}\Big[H_n\ (x-2j\ell-x_o,at) + \sum_{k=o}^{n} f_k\ H_k\ (x-(2j+1)\ell,at)$$
$$+ (-1)^n\ H_n\ (x-(2j+2)\ell+x_o,at)\Big]$$
$$- \sum_{j=o}^{N}\Big[H^*_n\ (x+2j\ell+x_o,at) + \sum_{k=o}^{n} f_k\ H^*_k\ (x+(2j+1)\ell,at)\Big]$$
$$+ (-1)^n\ H^*_n\ (x+(2j+2)\ell-x_o,at)\Big] \qquad (6.11)$$

The accuracy of u_N as an approximation for u in the domain $[0,\ell]$ x $[0,T]$ can be estimated by comparing the values of u_N along x=0, t=0 and x=ℓ with the prescribed values for u. We have

$$u_N(x,t) = \begin{cases} 0 & x=0 \\ h_n\ (x-x_o) & t=0 \\ H^*_n\ ((2N+1)\ell+x_o,at) + \sum_{k=o}^{n} f_k\ H^*_k\ ((2N+2)\ell,at) & \\ \quad + (-1)^n\ H^*_n\ ((2N+3)\ell-x_o,at), & x=\ell \end{cases}$$

It is clear that u_n satisfies the boundary and initial data exactly along x=0 and t=0 but does not vanish identically along x=ℓ, as required. However using (4.16) the truncated series can be bounded along x=ℓ in the form

$$|u_n(\ell,t)| \leq \text{const. exp}\left[-(x_o+(2N+1)\ell)^2/4at\right].$$

Since u and u_N are both solutions of (6.3) then the maximum principle for parabolic differential equations[9] states that their maximum difference occurs on the boundary. Thus the solution u of $(BVP)_3$ can be expressed in the form

$$u(x,t) = u_N(x,t) + R_N(x,t) \tag{6.12}$$

where R_n is a remainder term satisfying

$$|R_N(x,t)| \leq \text{const. } \exp\left[-(x_o+(2N+1)\ell)^2/4at\right]$$

uniformly in $[o,\ell] \times [o,T]$. The approximation u_N for u is therefore particularly accurate for small values of t. In fact for many applications it is sufficient to use only the lowest order approximation $u_o(x,t)$.

A similar analysis applys to problem $(BVP)_4$ which has the equivalent initial value problem

$$\left[D_t - aD_x^2\right] w = 0$$

$$w(x,0) = 2a^{-\gamma} \sum_{j=o}^{\infty} \left[h_{2\gamma}^*(x+2j\ell) - h_{2\gamma}(x-(2j+2)\ell)\right] \tag{IVP$_4$}$$

with the formal solution

$$w(x,t) = 2a^{-\gamma} \sum_{j=o}^{\infty} \left[H_{2\gamma}^*(x+2j\ell,at) - H_{2\gamma}(x-(2j+2)\ell,at)\right]. \tag{6.13}$$

In this form it is apparent that w(x,t) is antisymmetric and thus vanishes across $x=\ell$. On the other hand, by rearranging the terms into the equivalent form

$$w(x,t) = 2a^{-\gamma} H_{2\gamma}^*(x,at) + 2a^{-\gamma} \sum_{j=1}^{\infty} \left[H_{2\gamma}^*(x+2j\ell,at) - H_{2\gamma}(x-2j\ell,at)\right] \tag{6.14}$$

it becomes clear that w(x,t) satisfies the prescribed boundary condition along x=0. Denoting the truncated series by

$$w_N(x,t) = 2a^{-\gamma} \sum_{j=o}^{N} \left[H_{2\gamma}^*(x+2j\ell,at) - H_{2\gamma}(x-(2j+2)\ell,at)\right] \tag{6.15}$$

we can show as in the previous case that

$$w(x,t) = w_N(x,t) + \overline{R}_N(x,t) \tag{6.16}$$

[9] *A. N. Tikhonov and A. A. Samarskii, "Equations of Mathematical Physics", Page 206, Pergamon Press, Inc., MacMillan Company, New York, 1963.*

where

$$|\overline{R}_N(x,t)| \le \text{const} \exp\left[-((N+2)\ell)^2/4at\right]$$

uniformly in $[0,\ell] \times [0,T]$.

To conclude this section let us briefly touch on two problems where the function derivative (flux) values are specified at the boundary. Consider

$$\left[D_t - aD_x^2\right] u = 0 \qquad (BVP)_5$$

$$u(x,0) = h_\gamma(x-x_o)$$

$$u_x(0,t) = 0$$

$$u_x(\ell,t) = 0$$

and

$$\left[D_t - aD_x^2\right] w = 0 \qquad (BVP)_6$$

$$w(x,0) = 0$$

$$w_x(0,t) = h_\gamma(t)$$

$$W_x(\ell,t) = 0$$

By derivations similar to that for $(BVP)_3$ and $(BVP)_4$, only using symmetry in place of antisymmetry, we can arrive at the following solutions

$$\begin{aligned} u(x,t) = &\sum_{j=o}^{\infty}\Big[H_n(x-2j\ell-x_o,at) - \sum_{k=o}^{n} f_k\, H_k(x-(2j+1)\ell,at) \\ &\quad -(-1)^n H_n(x-(2j+2)\ell+x_o,at)\Big] \\ &+\sum_{j=o}^{\infty}\Big[H_n^*(x+2j\ell+x_o,at) - \sum_{k=o}^{n} f_k\, H^*_k(x+(2j+1)\ell,at) \\ &\quad -(-1)^n H_n^*(x+(2j+2)\ell-x_o,at)\Big] \end{aligned} \qquad (6.17)$$

and

$$w(x,t) = 2a^{-\gamma}\sum_{j=o}^{\infty}\left[H^*_{2\gamma+1}(x+2j\ell,at) + H_{2\gamma+1}(x-(2j+2)\ell,at)\right] \qquad (6.18)$$

for $(BVP)_5$ and $(BVP)_6$ respectively.

We should mention that the conversion of initial-boundary value problem to equivalent Cauchy problems works only for the diffusion equation and not for more complicated parabolic equations without special modification. Nevertheless in these cases we can use the Diffusion Equation Solution Sequence method, alluded to above, to obtain approximations which are comparable with the first term of the expansions just derived. This technique has been discussed in detail for Dirichlet type of boundary conditions (function values specified) and in a later report will be applied to Robin's type or connective heat transfer boundary conditions, such as occur in gun barrels.

VII. CONCLUSION

The function H_γ, H^*_γ and v_n have been defined and their properties investigated. In particular they have been shown to be solutions of the diffusion equation with the special initial values

$$H_\gamma(x,0) = h_\gamma(x)$$

$$H^*_\gamma(x,0) = h^*_\gamma(x)$$

$$v_n(x,0) = x^n/n!$$

where

$$h_\gamma(x) = \begin{cases} 0 & x \leq 0 \\ x^\gamma/\gamma! & x > 0 \end{cases}$$

and

$$h^*_\gamma(x) = h_\gamma(-x)$$

The functions H_γ and H^*_γ were then used to develop the series expansions (6.10), (6.13), (6.17) and (6.18) for the solutions of initial-boundary value problems $(BVP)_3$ - $(BVP)_6$.

APPENDIX THE POLYNOMIALS $p_{o,n}$ AND $q_{o,n}$

n	$p_{o,n}(x) = \bar{v}_n(x)$
<o	o
0	1
1	x
2	$(x^2+2)/2!$
3	$(x^3+6x)/3!$
4	$(x^4+12x^2+12)/4!$
5	$(x^5+20x^3+60x)/5!$
6	$(x^6+30x^4+180x^2+120)/6!$
7	$(x^7+42x^5+420x^3+3360x^2+840x)/7!$
8	$(x^8+56x^6+840x^4+3360x^2+1680)/8!$
9	$(x^9+72x^7+1512x^5+10,080x^3+15,120x)/9!$
10	$(x^{10}+90x^8+2520x^6+25,200x^4+75,600x^2+30,240)/10!$
11	$(x^{11}+110x^9+3960x^7+55,440x^5+277,200x^3+332,640x)/11!$
12	$(x^{12}+132x^{10}+5940x^8+110,880x^6+831,600x^4+1,995,840x^2+665,280)/12!$

In general

$$p_{o,n}(x) = \bar{v}_n(x) = \begin{cases} 0 & n = -1,-2,\ldots \\ \displaystyle\sum_{k=o}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!k!} & n = 0,1,2,\ldots \end{cases}$$

n	$q_{o,n}(x)$
0	0
1	2
2	x
3	$2(x^2+4)/3!$
4	$2(x^3+10x)/4!$
5	$2(x^4+18x^2+32)/5!$
6	$2(x^5+28x^3+132x)/6!$
7	$2(x^6+40x^4+348x^2+384)/7$
8	$2(x^7+54x^5+740x^3+2232x)/8!$
9.	$2(x^8+70x^6+1380x^4+7800x^2+5568)/9!$
10	$2(x^9+88x^7+2352x^5+21{,}120x^3+45{,}744x)/10!$
11	$2(x^{10}+108x^8+3752x^6+48{,}720x^4+201{,}744x^2+111{,}360)/11!$
12	$2(x^{11}+130x^9+5688x^7+100{,}464x^5+666{,}384x^3+1{,}117{,}728x)/12!$

In general

$$q_{o,n}(x) = (2/n!) \sum_{k=o}^{[(n-1)/2]} a_{nk} \; x^{n-1-2k}$$

where for n=1,2,3,...

$$a_{nk} = \begin{cases} 1 & \text{for } k=o \\ a_{n-1,k} + 2(n-1)\, a_{n-2,k-1} & \text{for } k=1,2, \ldots [(n-1)/2] \end{cases}$$

n	$q_{o,n}(x)$
-1	1
-2	$-x/2$
-3	$(x^2-2)/(-2)^2$
-4	$(x^3-6x)/(-2)^3$
-5	$(x^4-12x^2+12)/(-2)^4$
-6	$(x^5-20x^3+60x)/(-2)^5$
-7	$(x^6-30x^4+180x^2-120)/(-2)^6$
-8	$(x^7-42x^5+420x^3-840x)/(-2)^7$
-9	$(x^8-56x^6+840x^4-3360x^2+1680)/(-2)^8$
-10	$(x^9-72x^7+1512x^5-10{,}080x^3+15{,}120x)/(-2)^9$
-11	$(x^{10}-90x^8+2520x^6-25{,}200x^4+75{,}600x^2-30{,}240)/(-2)^{10}$
-12	$(x^{11}-110x^9+3960x^7-55{,}440x^5+277{,}200x^3-332{,}640x)/(-2)^{11}$

In general

$$q_{o,-n}(x) = (-2)^{1-n} \sum_{k=o}^{[(n-1)/2]} \frac{(-1)^k (n-1)!}{(n-1-2k)!\ k!} x^{n-1-2k}$$

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